MATH 2050 C Lecture 12 (Feb 24)

[Problem Set 6 posted, due on Mar 4.]

Last time: Monotone Convergence Theorem

Subsequences (Textbook §3.4)

Def$^{\underline{n}}$: Let $(x_n)_{n \in \mathbb{N}}$ be a seq. of real numbers. Suppose $n_1 < n_2 < n_3 < \cdots\cdots$ is a strictly increasing sequence of natural numbers. THEN:

$$(x_{n_k})_{k \in \mathbb{N}} := (x_{n_1}, x_{n_2}, x_{n_3}, \ldots, x_{n_k}, \ldots)$$

is called a subsequence of $(x_n)_{n \in \mathbb{N}}$

k^{th}-term in (x_{n_k})
$\shortparallel$
n_k^{th}-term in (x_n)

Picture:

$n \in \mathbb{N}$ $\quad (x_n) = (x_1, x_2, x_3, x_4, x_5, x_6, \ldots)$

$k \in \mathbb{N}$ $\quad (x_{n_k}) = (x_1, x_2, x_4, x_6, \ldots)$

$k=1 \quad k=2 \quad k=3 \quad k=4$

$n_1 = 1 \quad n_2 = 2 \quad n_3 = 4 \quad n_4 = 6$

<u>Example</u>: (Tail of a seq.) For each $\ell \in \mathbb{N}$, the ℓ-tail of a seq. $(x_n)_{n\in\mathbb{N}}$ is a subseq. $(x_{k+\ell})_{k\in\mathbb{N}}$

i.e $n_k := k + \ell$

<u>Example</u>: $(x_n) = ((-1)^n) = (-1, 1, -1, 1, -1, 1, \dots)$

$\rightsquigarrow$ Subseq. $(x_{2k}) = (1, 1, 1, 1, 1, \dots)$

<u>Idea</u>: Try to understand the convergence/divergence of the original seq. (x_n) by looking at its subsequences.

<u>Thm</u>: Suppose $\boxed{\lim_{n\to\infty} x_n = x}$. THEN, <u>every</u> subseq. (x_{n_k}) of (x_n) also converges to the same limit.

$\left(\text{i.e. } \lim_{n\to\infty} x_n = x \implies \lim_{k\to\infty} x_{n_k} = x \right)$

<u>Proof</u>: <u>Note</u>: $n_k \geq k \quad \forall k \in \mathbb{N}$ (Pf by induction)

Let $\varepsilon > 0$ be fixed but arbitrary.

$\lim_{n\to\infty} x_n = x \implies \exists K \in \mathbb{N}$ s.t $|x_n - x| < \varepsilon \quad \forall n \geq K$

Whenever $k \geq K$, we have $n_k \geq k \geq K$. hence

$$|x_{n_k} - x| < \varepsilon \quad \forall k \geq K$$

$\blacksquare$

Example (revisited) $\boxed{\lim_{n\to\infty} c^{\frac{1}{n}} = 1}$ where $c > 1$ is fixed

Pf: Let $(x_n) = (c^{\frac{1}{n}})$. One can show, by induction,

(x_n) is decreasing and bdd below by 1

$\overset{\text{MCT}}{\Longrightarrow}$ $x := \lim_{n\to\infty} x_n$ exists.

Consider only the even terms, i.e. the subseq.

$$(x_{n_k}) = (x_{2k})_{k\in\mathbb{N}} \quad \text{of} \quad (x_n)_{n\in\mathbb{N}}.$$

By Thm above, $\lim_{k\to\infty} x_{n_k} = x$

Observe: $x_{2n} := c^{\frac{1}{2n}} = (c^{\frac{1}{n}})^{\frac{1}{2}} = (x_n)^{\frac{1}{2}}$

Take $n\to\infty$ on both sides, we obtain

$$x = \sqrt{x} \Rightarrow x = \cancel{0} \text{ or } 1$$

rejected
$\because x_n \geq 1 \ \forall n$

$\blacksquare$

Remark: The theorem above also provides a "divergence criteria".

Thm: (x_n) convergent $\Rightarrow$ ANY subseq (x_{n_k}) of (x_n) converges to the SAME limit.

Cor: If any of the following holds:

(i) $\exists$ subseq. (x_{n_k}) which is divergent

(ii) $\exists$ two subseq, (x_{n_k}) & $(x_{n_{k'}})$ s.t

$$\lim_{k\to\infty} x_{n_k} \neq \lim_{k\to\infty} x_{n_{k'}}$$

then (x_n) is divergent.

E.g.) $((-1)^n)$ is divergent since $\exists$ two subseq.

$$(-1,-1,-1,-1,\dots) \to -1$$
$$\neq$$
$$(1,1,1,1,\dots) \to 1$$

E.g.) $(x_n) = (0,1,0,2,0,3,0,4,\dots,0,n,\dots)$
is divergent, since $\exists$ subseq

$(1,2,3,4,\dots,n,\dots)$ unbdd $\Rightarrow$ divergent